Class of colliding plane waves in terms of Jacobi functions

Nora Bretón[⋆*], Alberto García[⋆¶†], Alfredo Macías[⋄‡], and Gustavo Yáñez[⋆§]

[⋆] Departamento de Física,

CINVESTAV–IPN, Apartado Postal 14–740, C.P. 07000, México, D.F., MEXICO

[¶] Departamento de Física,

Universidad de Santiago de Chile,

Avda. Ecuador 3493–Casilla 307.Correo 2, Santiago, CHILE

[⋄] Departamento de Física,

Universidad Autónoma Metropolitana–Iztapalapa,

Apartado Postal 55–534, C.P. 09340, México, D.F., MEXICO.

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Abstract

We present a general class of noncolinear colliding wave solutions of the Einstein–Maxwell equations given in terms of fourth order polynomials, which in turn can be expressed through Jacobi functions depending on generalized advanced and retarded time coordinates. The solutions are characterized by six free parameters. The parameters can be chosen in such a way to avoid the generic focusing singularity.

[*]E-mail: nora@fis.cinvestav.mx

[†]E-mail: agarciad@lauca.usach.cl

[‡]E-mail: amac@xanum.uam.mx

[§]E-mail: gyanez@fis.cinvestav.mx

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I. INTRODUCTION

Recently, the collision of plane–fronted gravitational waves possibly coupled with electromagnetic waves, has been extensively studied [1–4]. Because gravity is always attractive, it was expected that focusing of the waves would occur and one of the interesting questions is how much focusing does general relativity predict. Within this framework, strong focusing would appear by the development of spacetime curvature singularities. Many solutions has been presented so far, describing collisions of plane–fronted gravitational and electromagnetic waves. And quite a few of them do develop Cauchy horizons. It is important to stress the fact that all of these solutions contain *only second degree* polynomials and that till now nobody has gone ahead to *higher degree polynomials* in the sense we are going to explain below.

All known colliding wave solutions are characterized, in null coordinates, by the line element

$$ds^2 = e^{-M} du dv + g_{ab} dx^a dx^b, \quad a, b = 1, 2, \tag{1.1}$$

where x^1, x^2 are ignorable coordinates and metric functions depending on (u, v) [5]. When written in (η, μ) coordinates [3], where η is a measure of the time from the instant of collision and μ is a measure of the distance between the plane wavefronts, the metric becomes

$$ds^2 = e^{2\gamma}(\eta^2 - \mu^2)\left(\frac{d\eta^2}{P(\eta)} - \frac{d\mu^2}{Q(\mu)}\right) + g_{ab} dx^a dx^b, \tag{1.2}$$

where the polynomials $P(\eta)$, $Q(\mu)$ are at most of the second degree in their respective variables. Generating techniques and other methods have allowed a great deal of generalizations in relation with the metric components g_{ab} (sometimes called Killingian sector) [6–13]. The generated solutions preserve the seed's structure of the polynomials $P(\eta), Q(\mu)$, changing in general the form of the metric function γ.

In this paper, we pursue further and present solutions with $P(\eta)$ and $Q(\mu)$ being polynomials of fourth degree, by means of a generalization of the advanced and retarded coordinates

concept. This generalization endows the corresponding spacetimes with a geometric structure for the time and independently for the spatial coordinate richer than the already known colliding wave solutions. In the limit of polynomials of second degree one recovers the widely used retarded and advanced time coordinates (u, v). The main purpose of this paper is to give the formulation of the most general class nowadays known of colliding waves with fourth degree polynomials. As usual, it is assumed that in the corresponding spacetime, the two waves approach each other from opposite sides in flat Minkowski background; after the collision, a new gravitational field evolves, which satisfies certain continuity conditions. The colliding plane waves possess five symmetries, while the geometry resulting after the collision has two spacelike Killing vectors.

The plan of the paper is as follows: In Sec. II we revisited some generalities on colliding waves concept. In Sec. III the general representation through advanced and retarded time coordinates for fourth degree polynomials is introduced. In Sec. IV the class of colliding waves with fourth degree polynomials is presented. In Sec. V the results are discussed.

II. COLLIDING WAVES

As was stated previously, in this section we review briefly some generalities on electrovacuum colliding waves, introduced by Ernst et al. [8].

The set of colliding waves solutions is described by the line element

$$g = 2g(u,v)dudv + g_{ab}(u,v)dx^a dx^b\,, \qquad a, b = 1, 2\,, \tag{2.1}$$

where $x^1 = x$, $x^2 = y$ are ignorable coordinates. The domain of the coordinate charts consists of $(x, y) \in \mathbf{R}^2$ and $(u, v) \in \mathbf{R}^2$; it is the union of four continuous regions: $I := \{(u, v) : 0 \leq u < 1, 0 \leq v < 1\}$, $II := \{(u, v) : u \leq 0, 0 \leq v < 1\}$, $III := \{(u, v) : 0 \leq u < 1, v < 0\}$, $IV := \{(u, v) : u \leq 0, v \leq 0\}$.

In the region IV, a closed subregion of the Minkowski space, it is required that

$$g_{\mu\nu}(u,v) = g_{\mu\nu}(0,0)\,, \quad A_\mu(u,v) = A_\mu(0,0)\,, \tag{2.2}$$

where A_μ is the electromagnetic vector potential. By scaling–shifting transformations the metric can be brought to standard Minkowski metric. In region II, the metric components and the electromagnetic vector potential depend only on v i.e. $g_{\mu\nu} = g_{\mu\nu}(0,v)$, $A_\mu = A_\mu(0,v)$. In region III these fields are functions of the coordinate u, i.e. $g_{\mu\nu} = g_{\mu\nu}(u,0)$, $A_\mu = A_\mu(u,0)$. In region I, which is occupied by the scattered null fields, the metric components and the electromagnetic field are functions of both u and v coordinates. In this way, the problem is reduced to know the metric (coframe) and the electromagnetic vector fuction A_μ.

A. Newman-Penrose tetrad

We introduce a complex null frame according to the conventional description:

$$e_\alpha = (\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}}) \tag{2.3}$$

with

$$\mathbf{l}^2 = \mathbf{n}^2 = \mathbf{m}^2 = \bar{\mathbf{m}}^2 = \mathbf{l}\cdot\mathbf{m} = \mathbf{n}\cdot\mathbf{m} = \mathbf{l}\cdot\bar{\mathbf{m}} = \mathbf{n}\cdot\bar{\mathbf{m}} = 0\,, \tag{2.4}$$

$$\mathbf{l}\cdot\mathbf{n} = 1, \quad \mathbf{m}\cdot\bar{\mathbf{m}} = -1\,. \tag{2.5}$$

The basis of the corresponding 1–forms, the coframe, is denoted by ϑ^α. If we lower the coframe index, then we use the same letters as with the frame: $\vartheta_\alpha = (\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}})$. Then the coframe reads

$$\vartheta^\alpha = (\mathbf{n}, \mathbf{l}, -\bar{\mathbf{m}}, -\mathbf{m})\,. \tag{2.6}$$

According to (2.4) and (2.5), we have

$$g = 2\left(\vartheta^{\hat{0}} \otimes \vartheta^{\hat{1}} - \vartheta^{\hat{2}} \otimes \vartheta^{\hat{3}}\right) = n_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^\beta\,, \tag{2.7}$$

with

$$n_{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} . \tag{2.8}$$

B. Self–dual part of the complex Weyl 2–form

Quite independently from the type of coframe we are using, we can consider some properties of the Weyl 2–form. Let us denote the Weyl curvature 2–form by

$$C_{\alpha\beta} = \frac{1}{2}\, C_{\mu\nu\alpha\beta}\, \vartheta^{\mu} \wedge \vartheta^{\nu} . \tag{2.9}$$

If, in four dimensions, an arbitrary complex 2–form ω is given, then its Hodge dual ${}^{\star}\omega$ is again a 2-form. Therefore we can build linear combinations of both:

$$\omega^{\pm} := \frac{1}{2}\left(\omega \pm i\, {}^{\star}\omega\right) . \tag{2.10}$$

Here i is the imaginary unit. We call ω^{+} the self–dual and ω^{-} the antiself–dual part of the 2–form ω. Clearly,

$$\omega = \omega^{+} + \omega^{-} . \tag{2.11}$$

A form ω is called self–dual, if its antiself–dual piece vanishes, i.e., if $\omega = i\, {}^{\star}\omega$, it is called antiself–dual, if $\omega = -i\, {}^{\star}\omega$.

Let us now assume that the form ω is *real*, that is, $\omega = \overline{\omega}$, where the bar denotes complex conjugation. Then, according to (2.10), they fulfill the relations:

$$\overline{\omega^{+}} = \omega^{-} , \qquad \text{and} \qquad \overline{\omega^{-}} = \omega^{+} . \tag{2.12}$$

Consequently, a real 2–form $\omega = \overline{\omega}$ can be alternatively encoded into its self–dual part ω^{+}. Incidentally, the same is true for its antiself–dual part.

The Weyl 2–form is a real quantity. Accordingly, we can also take its self–dual part

$$C^{+}_{\alpha\beta} := \frac{1}{2}\left(C_{\alpha\beta} + i\,{}^{\star}C_{\alpha\beta}\right) . \qquad (2.13)$$

Expanding the Weyl 2–form $C^{+}_{\alpha\beta}$ in terms of the basis $\vartheta^{\mu} \wedge \vartheta^{\nu}$ we have

$$C^{+}_{\alpha\beta} = \frac{1}{2}\, C^{+}_{\mu\nu\alpha\beta}\, \vartheta^{\mu} \wedge \vartheta^{\nu} , \qquad (2.14)$$

which we will asumme to be in the standard 2–form selfdual basis $[\mathbf{U}, \mathbf{V}, \mathbf{W}]$. The most general form of our Weyl field components, compatible with colliding wave spacetime structure is given by [14]:

$$\begin{aligned}
&A_{\mu} = A_{\mu}(u, v),\ C^{+}_{abcd} = 2\Psi_0 U_{ab}U_{cd} + 2\Psi_2(U_{ab}V_{cd} + V_{ab}U_{cd} + W_{ab}W_{cd}) + 2\Psi_4 V_{ab}V_{cd}\, , \text{region I},\\
&A_{\mu} = A_{\mu}(v)\, , \quad C^{+}_{abcd} = 2\Psi_0 U_{ab}U_{cd}\, , \quad \text{region II}\, ,\\
&A_{\mu} = A_{\mu}(u)\, , \quad C^{+}_{abcd} = 2\Psi_4 V_{ab}V_{cd}\, , \quad \text{region III}\, , \qquad (2.15)
\end{aligned}$$

with

$$\begin{aligned}
W_{ab} &= m_a \tilde{m}_b - m_b \tilde{m}_a - k_a l_b + k_b l_a\, ,\\
V_{ab} &= k_a m_b - k_b m_a\, ,\\
U_{ab} &= -l_a \tilde{m}_b + l_b \tilde{m}_a\, . \qquad (2.16)
\end{aligned}$$

where $m_a, \tilde{m}_b, k_a$ and l_a are null tetrads.

III. GENERALIZED ADVANCED AND RETARDED TIME COORDINATES

We shall consider a metric of the form

$$g = e^{-2\gamma}\left(\frac{dp^2}{P(p)} - \frac{dq^2}{Q(q)}\right) + g_{ab}dx^a dx^b,\ , \qquad (3.1)$$

where P, Q are fourth degree polynomials in their respective variable, and γ depending on both (p, q) coordinates.

Since we have real polynomials of fourth degree, when looking for advanced and retarded time variables one has to deal with elliptic integrals, i.e. we shall use the Legendre first kind integrals:

$$\int \frac{dr}{\sqrt{G_4(r)}} = \mu \int \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \mu F(\phi, k)\,, \tag{3.2}$$

where the fourth degree polynomial $G_4(r)$, is represented by the product of four monomials containing the corresponding roots, the coefficient of the higher degree (fourth), denoted by a_4, is assumed to be $a_4 = \pm 1$; if this were no the case, i. e. $a_4 = \pm\alpha_4, \alpha_4 > 0$, then dividing by α_4 one arrives at the above equation where now μ has to be interpreted as $\frac{\mu}{\sqrt{\alpha_4}}$, in what follows we shall adopt these conventions when needed. The real roots are denoted by r_j, with $j = 1, \ldots, 4$ and $r_1 > r_2 > r_3 > r_4$, while the complex roots by $s_1 \pm it_1$, $s_2 \pm it_2$, with $s_1 \geq s_2$, $t_1 > t_2 > 0$, we introduce also the following useful notation

$$\begin{aligned}
r_{ik} = r_k - r_i, &\quad (i, k = 1, 2, 3, 4)\,,\\
(r, \beta, \gamma, \delta) &= \frac{r-\gamma}{r-\delta}\frac{\beta-\delta}{\beta-\gamma}\,,\\
\tan\theta_1 &= \frac{r_1 - s_1}{t_1}\,, \quad \tan\theta_2 = \frac{r_2 - s_1}{t_1}\,,\\
\tan\theta_3 &= \frac{t_1 + t_2}{s_1 - s_2}\,, \quad \tan\theta_4 = \frac{t_1 - t_2}{s_1 - s_2}\,,\\
\tan\left[(\theta_5/2)^2\right] &= \frac{\cos\theta_3}{\cos\theta_4}\,,\\
\nu = \tan\left[(\theta_2 - \theta_1)/2\right] &\tan\left[(\theta_2 + \theta_1)/2\right]\,.
\end{aligned} \tag{3.3}$$

The elliptic integral $F(\phi, k)$ is the Legendre integral of the first kind. As it is well known, the standard form for writing this function is

$$z = \int^{\phi} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = F(\phi, k)\,, \quad \phi = \mathrm{am}\, z\,, \tag{3.4}$$

where $\mathrm{am}\, z$ denotes the function amplitude of z. Replacing ϕ through ω according to

$$\omega = \sin\phi = \sin(\mathrm{am}\, z) := \mathrm{sn}\, z\,, \quad z = \int_0^{\omega} \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}} = \widetilde{F}(\omega, k)\,, \tag{3.5}$$

where $\mathrm{sn}\, z$ belongs to the Jacobi family of elliptic functions $(\mathrm{sn}\, z, \mathrm{cn}\, z, \mathrm{dn}\, z)$, with well established analytical properties. The values and main properties of these function can be found in [15].

Going back to the line element (3.1) and assuming that $P(p)$ and $Q(q)$ are polynomials up to fourth degree on p and q respectively, then the two dimensional line element, first two

terms in (3.1), can be written in terms of the retarded and advanced time coordinates $\widetilde{u}$ and $\widetilde{v}$, respectively, as follows

$$e^{-2\gamma(p,q)}\left(\frac{dp^2}{P(p)}-\frac{dq^2}{Q(q)}\right)=e^{-\gamma(\widetilde{u},\widetilde{v})}d\widetilde{u}d\widetilde{v}\,, \tag{3.6}$$

with

$$\left.\begin{array}{c} d\widetilde{u} \\ \\ d\widetilde{v} \end{array}\right\} = \frac{dp}{\sqrt{P_4(p)}} \pm \frac{dq}{\sqrt{Q_4(q)}} = \frac{\mu_\phi}{\sqrt{\alpha_4}}\frac{d\phi}{\sqrt{1-k_\phi^2\sin^2\phi}} \pm \frac{\mu_\theta}{\sqrt{\beta_4}}\frac{d\theta}{\sqrt{1-k_\theta^2\sin^2\theta}}\,, \tag{3.7}$$

where μ_ϕ, μ_θ are constants and α_4, β_4 are the moduli of the coefficients of the higher power in the polynomials P and Q, correspondingly. The variables $\widetilde{u}$ and $\widetilde{v}$ range the values from $-\infty$ to $+\infty$.

The integration of (3.7) yields

$$\left.\begin{array}{c} \widetilde{u} \\ \\ \widetilde{v} \end{array}\right\} = \frac{\mu_\phi}{\sqrt{\alpha_4}}F(\phi,k_\phi) \pm \frac{\mu_\theta}{\sqrt{\beta_4}}F(\theta,k_\theta)\,, \tag{3.8}$$

where $-\infty < \widetilde{u} < \infty$, $-\infty < \widetilde{v} < \infty$. Hence, one has relations between the Legendre elliptic integrals and the null coordinates $\widetilde{u}$, $\widetilde{v}$, i.e.

$$F(\phi,k_\phi) = \frac{\sqrt{\alpha_4}}{2\mu_\phi}\left(\widetilde{u}+\widetilde{v}\right)\,, \quad F(\theta,k_\theta) = \frac{\sqrt{\beta_4}}{2\mu_\theta}\left(\widetilde{u}-\widetilde{v}\right)\,. \tag{3.9}$$

Consequently, the inversion formulas bring the following functional dependence upon the generalized advanced and retarded time coordinates

$$\phi = \mathrm{am}\left[\frac{\sqrt{\alpha_4}}{2\mu_\phi}\left(\widetilde{u}+\widetilde{v}\right)\right]\,, \quad \omega = \sin\phi = \mathrm{sn}\left[\frac{\sqrt{\alpha_4}}{2\mu_\phi}\left(\widetilde{u}+\widetilde{v}\right)\right]\,, \tag{3.10}$$

$$\theta = \mathrm{am}\left[\frac{\sqrt{\beta_4}}{2\mu_\theta}\left(\widetilde{u}-\widetilde{v}\right)\right]\,, \quad \Omega = \sin\theta = \mathrm{sn}\left[\frac{\sqrt{\beta_4}}{2\mu_\theta}\left(\widetilde{u}-\widetilde{v}\right)\right]\,, \tag{3.11}$$

which allow us to give the original p and q coordinates, initially expressed through ϕ and θ, in terms of $\widetilde{u}$ and $\widetilde{v}$ null coordinates, i.e.

$$p = p(\widetilde{u},\widetilde{v})\,, \quad \text{and} \quad q = q(\widetilde{u},\widetilde{v})\,. \tag{3.12}$$

Moreover, one can introduce a new set of variables (u, v)

$$u = \sin\left(\frac{\widetilde{u}}{2}\right), \quad \text{and} \quad v = \sin\left(\frac{\widetilde{v}}{2}\right). \tag{3.13}$$

where the ranges of the variables are $-\infty < \widetilde{u} < \infty$, $-\infty < \widetilde{v} < \infty$ and $-1 < u < 1, \quad -1 < v < 1$. These variables are such that

$$d\widetilde{u} = 2\frac{du}{U}, \quad d\widetilde{v} = 2\frac{dv}{V} \tag{3.14}$$

and

$$\frac{\widetilde{u} \pm \widetilde{v}}{2} = \arcsin u \pm \arcsin v = \arcsin[uV \pm vU] \tag{3.15}$$

where $U = \sqrt{1-u^2}, V = \sqrt{1-v^2}$ and the corresponding term in the line element

$$d\widetilde{u} \otimes d\widetilde{v} = 4\frac{du}{U} \otimes \frac{dv}{V}. \tag{3.16}$$

Having in mind colliding wave interpretations of the results, one restricts u and v to the range $0 \leq u < 1, 0 \leq v < 1$ to be closed to the standard colliding wave approach. Depending on the character, real or complex, of the roots of the fourth degree polynomials, one may encounter in general combinations of different possible cases. It is always possible to introduce the retarded and advanced time coordinates u and v. However, only certain solutions satisfy the Ernst [8] requirements for colliding waves. We shall restrict ourselves to metrics which can support a colliding wave interpretation.

It may occur several mixed cases depending on the degree of the polynomials appearing in the non-killingian sector, for instance,

$$\frac{dp^2}{P_4} - \frac{dq^2}{Q_2}, \quad \frac{dp^2}{P_3} - \frac{dq^2}{Q_2}, \tag{3.17}$$

where the subindice indicates the degree of the polynomial we are dealing with; the possibility of interchanging P by Q and p by q and viceversa has to be taken also into account. The case $\frac{dp^2}{P_3} - \frac{dq^2}{Q_3}$ can be treated in the manner we did with the fourth degree polynomial.

Let us consider the case of $\frac{dp^2}{P_4} - \frac{dq^2}{Q_2}$. Introducing the null variables $(\widetilde{u}, \widetilde{v})$,

$$\left.\begin{matrix} d\widetilde{u} \\ d\widetilde{v} \end{matrix}\right\} = \frac{dp}{\sqrt{P_4(p)}} \pm \frac{dq}{\sqrt{Q_2(q)}}, \tag{3.18}$$

in the case when Q_2 can be brought to the form $1-q^2$, they become

$$\left.\begin{array}{c}\widetilde{u}\\ \\ \widetilde{v}\end{array}\right\} = \mu_\phi F(\phi, k_\phi) \pm \arcsin q, \ , \tag{3.19}$$

therefore

$$\frac{\widetilde{u}-\widetilde{v}}{2} = \arcsin q, \tag{3.20}$$

$$\frac{\widetilde{u}+\widetilde{v}}{2} = \mu_\phi F(\phi, k_\phi) = \mu_\phi \int^\phi \frac{dx}{\sqrt{1-k_\phi^2 \sin^2 x}} = \mu_\phi z, \quad \phi = \mathrm{am} z, \tag{3.21}$$

hence

$$q = \sin(\frac{\widetilde{u}-\widetilde{v}}{2}), \quad p = p(\frac{\widetilde{u}+\widetilde{v}}{2}) \tag{3.22}$$

where the relation of p in terms of $\widetilde{u}, \widetilde{v}$ depend on the roots of the polynomial P_4. The other case with P_3, Q_2 can be treated in similar way.

For completeness we also briefly comment the following cases

(i)

$$\frac{dp^2}{P_{4,3}(p)} - \frac{dq^2}{Q_1(q)} \rightarrow \frac{dp^2}{P_{4,3}(p)} - \beta(dq^{\frac{1}{2}})^2, \quad q > 0, \beta > 0, \tag{3.23}$$

the corresponding $\widetilde{u}, \widetilde{v}$ are

$$\left.\begin{array}{c}\widetilde{u}\\ \\ \widetilde{v}\end{array}\right\} = \mu_\phi F(\phi, k_\phi) \pm \beta q^{\frac{1}{2}}, \tag{3.24}$$

thus

$$p = p(\frac{\widetilde{u}+\widetilde{v}}{2}), \quad q = (\frac{\widetilde{u}-\widetilde{v}}{2\beta})^2, \tag{3.25}$$

(ii)

$$\frac{dp^2}{P_2(p)} - \frac{dq^2}{Q_2(q)} \rightarrow \frac{dp^2}{1-p^2} - \frac{dq^2}{1-q^2}, \tag{3.26}$$

the corresponding $\widetilde{u}, \widetilde{v}$ are

$$\left.\begin{array}{c}\widetilde{u}\\ \\ \widetilde{v}\end{array}\right\}=\arcsin p\pm\arcsin q, \tag{3.27}$$

hence

$$p=\sin(\frac{\widetilde{u}+\widetilde{v}}{2}),\quad q=\sin(\frac{\widetilde{u}-\widetilde{v}}{2}), \tag{3.28}$$

(iii)

$$\frac{dp^2}{P_{4,3}(p)}-dq^2, \tag{3.29}$$

the corresponding $\widetilde{u},\widetilde{v}$ are

$$\left.\begin{array}{c}\widetilde{u}\\ \\ \widetilde{v}\end{array}\right\}=\mu_\phi F(\phi,k_\phi)\pm q, \tag{3.30}$$

hence

$$p=p(\frac{\widetilde{u}+\widetilde{v}}{2}),\quad q=\frac{\widetilde{u}-\widetilde{v}}{2}, \tag{3.31}$$

(iv)

$$\frac{dp^2}{P_2(p)}-\frac{dq^2}{Q_1}\to\frac{dp^2}{1-p^2}-\beta(dq^{\frac{1}{2}})^2,\quad \beta>0,q>0, \tag{3.32}$$

the corresponding $\widetilde{u},\widetilde{v}$ are

$$\left.\begin{array}{c}\widetilde{u}\\ \\ \widetilde{v}\end{array}\right\}=\arcsin p\pm\beta q^{\frac{1}{2}}, \tag{3.33}$$

hence

$$p=\sin(\frac{\widetilde{u}+\widetilde{v}}{2}),\quad q=(\frac{\widetilde{u}-\widetilde{v}}{2\beta})^2, \tag{3.34}$$

(v)

$$\frac{dp^2}{P_1(p)}-\frac{dq^2}{Q_1(q)}\to(dp^{\frac{1}{2}})^2-(dq^{\frac{1}{2}})^2,\quad p>0,q>0, \tag{3.35}$$

the corresponding $\widetilde{u},\widetilde{v}$ are

$$\left.\begin{array}{c}\widetilde{u}\\ \\ \widetilde{v}\end{array}\right\} = p^{\frac{1}{2}} \pm q^{\frac{1}{2}}, \tag{3.36}$$

thus

$$p = (\frac{\widetilde{u}+\widetilde{v}}{2})^2, \quad q = (\frac{\widetilde{u}-\widetilde{v}}{2})^2, \tag{3.37}$$

A. Different transformations of $p(\widetilde{u},\widetilde{v})$ and $q(\widetilde{u},\widetilde{v})$.

In this subsection, we present all possible cases depending upon the roots of the polynomials $P(p)$ and $Q(q)$ and the corresponding transformation.

1. For P and Q with all real and different roots, i.e.

$$P = \alpha_4 \epsilon_\pm \left[p-p_1\right]\left[p-p_2\right]\left[p-p_3\right]\left[p-p_4\right], \qquad \epsilon_\pm = \pm 1,\ \alpha_4 > 0\,, \tag{3.38}$$

$$Q = \beta_4 \nu_\pm \left[q-q_1\right]\left[q-q_2\right]\left[q-q_3\right]\left[q-q_4\right], \qquad \nu_\pm = \pm 1,\ \beta_4 > 0\,, \tag{3.39}$$

The corresponding transformations read as follows

$P_4(p)$	$\epsilon_\pm$	Interval	$p(\widetilde{u},\widetilde{v}) =$	k^2
four	ϵ_+	$p_1 \leq p$ or $p \leq p_4$	$\frac{p_1 p_{42} - p_2 p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{p_{42} - p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	
real	ϵ_+	$p_3 \leq p \leq p_2$	$\frac{p_3 p_{42} - p_4 p_{32}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{p_{42} - p_{32}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	$(\alpha_1,\alpha_2,\alpha_4,\alpha_3)$
roots	ϵ_-	$p_4 \leq p \leq p_3$	$\frac{p_4 p_{31} + p_1 p_{43}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{p_{31} + p_{43}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	
	ϵ_-	$p_2 \leq p \leq p_1$	$\frac{p_2 p_{31} - p_3 p_{21}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{p_{31} - p_{21}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	$(\alpha_3,\alpha_2,\alpha_4,\alpha_1)$

In the transformations given above we must understand that $\widetilde{u} = 2\arcsin u$ and $\widetilde{v} = 2\arcsin v$, such that $\widetilde{u}\pm\widetilde{v} = 2\arcsin(uV \pm vU)$ and μ_ϕ has to be understood as $\mu_\phi/\sqrt{\alpha_4}$ when $\alpha_4 \neq 1$. For $Q(q)$ the same Table is valid making the changes: $p \to q$, $p_i \to q_i$,

$\epsilon_\pm \longrightarrow \nu_\pm$, $\mu_\phi \longrightarrow \mu_\theta$ and $\widetilde{u} + \widetilde{v} \longrightarrow \widetilde{u} - \widetilde{v}$, $s_i \longrightarrow \sigma_i$ and $t_i \longrightarrow \tau_i$. The same applies in the following cases.

2. For the case in which P and Q possess two real and different roots, and two complex roots, i.e.

$$P = \alpha_4 \epsilon_\pm \left[p - p_1\right] \left[p - p_2\right] \left[p - (s_1 + it_1)\right] \left[p - (s_1 - it_1)\right], \quad \alpha_4 > 0, \tag{3.40}$$

$$Q = \beta_4 \nu_\pm \left[q - q_1\right] \left[q - q_2\right] \left[q - (\widetilde{\sigma}_1 + i\widetilde{\tau}_1)\right] \left[q - (\widetilde{\sigma}_1 - i\widetilde{\tau}_1)\right], \quad \beta_4 > 0, \tag{3.41}$$

the transformations are given by

$P_4(p)$	$\epsilon_\pm$	*Interval*	$p(\widetilde{u},\widetilde{v}) =$	k^2
two complex and two real roots	ϵ_+	$-\infty < p < \infty$	$\frac{p_1+p_2}{2} - \frac{p_1-p_2}{2}\left(\frac{\nu_\theta - \mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})}{1-\nu_\theta \mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})}\right)$	$\left(\sin(\frac{\theta_1-\theta_2}{2})\right)^2$

3. In the case P and Q with four complex roots, respectively, i.e.

$$P = \alpha_4 \left[p - (s_1 + it_1)\right] \left[p - (s_1 - it_1)\right] \left[p - (s_2 + it_2)\right] \left[p - (s_2 - it_2)\right], \tag{3.42}$$

$$Q = \beta_4 \left[q - (\widetilde{\sigma}_1 + i\widetilde{\tau}_1)\right] \left[q - (\widetilde{\sigma}_1 - i\widetilde{\tau}_1)\right] \left[q - (\widetilde{\sigma}_2 + i\widetilde{\tau}_2)\right] \left[q - (\widetilde{\sigma}_2 - i\widetilde{\tau}_2)\right], \tag{3.43}$$

the transformations are now

$P_4(p)$	$\epsilon_\pm$	Interval	$p(\widetilde{u},\widetilde{v}) =$	k^2
$s_1 > s_2$	ϵ_+	$-\infty < p < \infty$	$s_1 + t_1\left[\frac{\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\cos\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)+\mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\sin\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)}{\mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\cos\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)-\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\sin\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)}\right]$	$\sin^2\theta_5$
$s_1 = s_2$ B $t_1 > t_2$			$s_1 - t_1\left[\frac{\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\cos\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)+\mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\sin\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)}{\mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\cos\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)-\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\sin\left(\frac{\theta_3}{2}+\frac{\theta_4}{2}\right)}\right]$	

We include the transformations for the case of polynomials of third degree:

$P_3(p)$	$\epsilon_\pm$	Interval	$p(\widetilde{u},\widetilde{v}) =$	k^2
three	ϵ_+	$p_3 \leq p \leq p_2$	$\frac{p_{32}}{p_3}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2$	$\frac{p_{32}}{p_{31}}$
real		$p_1 \leq p$	$\frac{p_1 - p_2\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{1-\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	
roots	ϵ_-	$p \leq p_1$	$\frac{p_1\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2 - p_{31}}{\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	$\frac{p_{21}}{p_{31}}$
		$p_2 \leq p \leq p_1$	$\frac{p_2 p_{31} - p_3 p_{21}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{p_{31} - p_{21}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}$	
two	ϵ_+	$p_1 \leq p$	$p_1 - \frac{c_1\left(1-\mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)}{\cos\theta_1\left(1-\mathrm{cn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)}$	$\left(\sin\left[\frac{\theta_1}{2}+\frac{\pi}{4}\right]\right)^2$
complex and one real	ϵ_-	$p \leq p_1$		

For the polynomial Q the same Tables are valid making the changes: $p \to q$, $p_i \to q_i$,$\epsilon_\pm \to \nu_\pm$, $\mu_\phi \to \mu_\theta$ and $\widetilde{u}+\widetilde{v} \to \widetilde{u}-\widetilde{v}$, $s_i \to \sigma_i$ and $t_i \to \tau_i$.

All possible combinations between the transformations corresponding to the different possible ranges of p and q should be taken into account. Moreover, it is straightforward to perform such combinations by using the explicit transformations in tables given above.

B. Limiting transition

For completeness we include the limiting procedure to obtain from the fourth order polynomials the third and second degree ones.

$$\begin{aligned} P &= \epsilon_\pm a_4(p-p_1)(p-p_2)(p-p_3)(p-p_4) \\ &= a_4 p_1(\frac{p}{p_1}-1)(p-p_2)(p-p_3)(p-p_4) \\ &= \alpha_4(\frac{p}{p_1}-1)(p-p_2)(p-p_3)(p-p_4), \end{aligned} \tag{3.44}$$

Taking the limit when $p_1 \to \infty$ we recover a third degree polynomial:

$$\lim_{p_1\to\infty} P_4 = \epsilon_\pm\alpha_4(p-p_2)(p-p_3)(p-p_4) = \epsilon_\pm\alpha_4 P_3, \tag{3.45}$$

Analogously we take the limit on P_3 when $p_2 \to \infty$ to get a second order polynomial:

$$\lim_{p_2\to\infty} P_3 = \lim_{p_2\to\infty}\left\{\epsilon_\pm a_4 p_2(\frac{p}{p_2}-1)(p-p_3)(p-p_4)\right\}$$
$$= \lim_{p_2\to\infty}\left\{\epsilon_\pm \alpha_3(\frac{p}{p_2}-1)(p-p_3)(p-p_4)\right\} = \epsilon_\pm\alpha_3(p-p_3)(p-p_4) = \epsilon_\pm\alpha_3 P_2, \quad (3.46)$$

The corresponding procedure can be achieved for Q_4 to obtain a second order polynomial.

C. Concrete example

As a concrete example let us consider, for instance, the case P with four real and different roots and Q also with four real and different roots, in which the coefficients of the higher degree are 1, and the roots of P and Q are denoted by p_i and q_i respectively. For the elliptic integral depending on the p–coordinate, with $P(p) = (p-p_1)(p-p_2)(p-p_3)(p-p_4)$, $(\epsilon_\pm = 1)$, one has the relation

$$\int \frac{dx}{\sqrt{P(p)}} = \mu_\phi \int \frac{d\phi}{\sqrt{1-k_\phi^2\sin^2\phi}} = \mu_\phi F(\phi, k)\,, \quad (3.47)$$

where the explicit relation between p and ϕ reads

$$p = \frac{p_1 p_{42} - p_2 p_{41}\sin^2\phi}{p_{42} - p_{41}\sin^2\phi}\,,$$
$$\sin^2\phi = \frac{p_{42}}{p_{41}}\left(\frac{p-p_1}{p-p_2}\right)\,, \quad \to \quad \phi = \arcsin\left(\pm\sqrt{\frac{p_{42}}{p_{41}}\left(\frac{p-p_1}{p-p_2}\right)}\right)\,. \quad (3.48)$$

The parameter k_ϕ, $0 < k_\phi^2 < 1$, is given by

$$k^2 = (p_1, p_2, p_4, p_3) = \left(\frac{p_1-p_4}{p_1-p_3}\right)\left(\frac{p_2-p_3}{p_2-p_4}\right)\,, \quad \text{and} \quad \mu = \frac{\mu_\phi}{\sqrt{\alpha_4}} = \frac{2}{\sqrt{p_{31}p_{42}}}\,. \quad (3.49)$$

It is straightforward to find the analogous expression for the elliptic integral depending on the q–coordinate. The explicit expression of the coordinates p and q through $\widetilde{u}$ and $\widetilde{v}$ is the following

$$p(\widetilde{u},\widetilde{v}) = \frac{p_1 p_{42} - p_2 p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}{p_{42} - p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}(\widetilde{u}+\widetilde{v})\right)^2}\,, \quad (3.50)$$

$$q(\widetilde{u},\widetilde{v}) = \frac{q_1 q_{42} - q_2 q_{41}\left(\mathrm{sn}\frac{1}{2\mu_\theta}(\widetilde{u}-\widetilde{v})\right)^2}{q_{42} - q_{41}\left(\mathrm{sn}\frac{1}{2\mu_\theta}(\widetilde{u}-\widetilde{v})\right)^2}\,. \quad (3.51)$$

In terms of the null variables u and v which bring the line element to the form

$$\frac{dp^2}{P_4} - \frac{dq^2}{Q_4} = d\widetilde{u}d\widetilde{v} = 4\frac{dudv}{UV} \tag{3.52}$$

the p and q variables amount to

$$p(u,v) = \frac{p_1p_{42} - p_2p_{41}\left(\mathrm{sn}_{\frac{1}{\mu_\phi}}(\arcsin(uV+vU))\right)^2}{p_{42} - p_{41}\left(\mathrm{sn}_{\frac{1}{2\mu_\phi}}(\arcsin(uV+vU))\right)^2}\,, \tag{3.53}$$

$$q(u,v) = \frac{q_1q_{42} - q_2q_{41}\left(\mathrm{sn}_{\frac{1}{2\mu_\theta}}(\arcsin(uV-vU))\right)^2}{q_{42} - q_{41}\left(\mathrm{sn}_{\frac{1}{2\mu_\theta}}(\arcsin(uV-vU))\right)^2}\,, \tag{3.54}$$

On the other hand, the relations between ϕ and θ with p and q are

$$\phi \pm \theta = \arcsin\sqrt{\frac{p_{42}}{p_{41}}\left(\frac{p-p_1}{p-p_2}\right)} \pm \arcsin\sqrt{\frac{q_{42}}{q_{41}}\left(\frac{q-q_1}{q-q_2}\right)}\,, \tag{3.55}$$

where ϕ and θ stand correspondingly for the arguments of the elliptical Legendre integrals related with the integration of P and Q. We define the auxiliary non-null variables u' and v' as

$$u' = \sin\left(\frac{\phi+\theta}{2}\right)\,, \quad v' = \sin\left(\frac{\phi-\theta}{2}\right)\,, \tag{3.56}$$

with

$$\left.\begin{matrix}\phi\\ \\ \theta\end{matrix}\right\} = \arcsin u' \pm \arcsin v' = \arcsin\left[u'\sqrt{1-v'^2} \pm v'\sqrt{1-u'^2}\right] = \arcsin\left[u'V' \pm v'U'\right]\,, \tag{3.57}$$

where

$$V' = \sqrt{1-v'^2}\,, \quad \text{and} \quad U' = \sqrt{1-u'^2}\,. \tag{3.58}$$

Thus, when we substitute (3.57) and (3.58) into (3.50) and (3.51) it is staightforward to find

$$p(u',v') = \frac{p_1p_{42} - p_2p_{41}\left(u'V'+v'U'\right)^2}{p_{42} - p_{41}\left(u'V'+v'U'\right)^2}\,, \tag{3.59}$$

$$q(u',v') = \frac{q_1q_{42} - q_2q_{41}\left(u'V'-v'U'\right)^2}{q_{42} - q_{41}\left(u'V'-v'U'\right)^2}\,. \tag{3.60}$$

Moreover, from (3.7) and (3.57), the advanced and retarded time coordinates $\widetilde{u}$ and $\widetilde{v}$ are related with the variables u' and v' through

$$\left.\begin{array}{c} d\widetilde{u} \\ \\ d\widetilde{v} \end{array}\right\} = \left[\frac{\mu_\phi}{\sqrt{1-k_\phi^2(u'V'+vU')^2}} \pm \frac{\mu_\theta}{\sqrt{1-k_\theta^2(u'V'-v'U')^2}}\right]\frac{du'}{U'}$$
$$+\left[\frac{\mu_\phi}{\sqrt{1-k_\phi^2(u'V'+v'U')^2}} \mp \frac{\mu_\theta}{\sqrt{1-k_\theta^2(u'V'-v'U')^2}}\right]\frac{dv'}{V'}\,. \tag{3.61}$$

According to our general procedure (3.61) can be inverted, yielding $u' = u'(\widetilde{u},\widetilde{v})$ and $v' = v'(\widetilde{u},\widetilde{v})$.

In the limit of polynomials of second degree , i.e. P_2, Q_2, one has $k_\phi = k_\theta = 0$, and $\mu_\phi, \mu_\theta \to 1$, hence one recovers from (3.61) the widely used retarded and advanced time coordinates

$$d\widetilde{u} = 2\frac{du'}{U'} \quad d\widetilde{v} = 2\frac{dv'}{V'}\,, \tag{3.62}$$

In the next Section we present an explicit class of colliding wave solutions.

IV. A CLASS OF COLLIDING WAVES

Let us consider an explicit solution of the Einstein-Maxwell equations in region I, given by the following coframe in coordinates (x,p,q,y):

$$\vartheta^{\hat{0}} = \frac{1}{H}\sqrt{\frac{\Delta}{Q}}\,dq, \quad \vartheta^{\hat{1}} = \frac{1}{H}\sqrt{\frac{Q}{\Delta}}\,(dx+p^2dy)\,,$$
$$\vartheta^{\hat{2}} = \frac{1}{H}\sqrt{\frac{\Delta}{P}}\,dp, \quad \vartheta^{\hat{3}} = \frac{1}{H}\sqrt{\frac{Q}{\Delta}}\,(dx-q^2dy)\,. \tag{4.1}$$

Here we have the functions $H = H(p,q)$, $P = P(p)$, $Q = Q(q)$, and $\Delta = \Delta(p,q)$. The coframe is assumed to be orthonormal

$$g = o_{\alpha\beta}\,\vartheta^\alpha \otimes \vartheta^b\,. \tag{4.2}$$

Then the metric explicitly reads

$$
g = \frac{1}{H^2} \left\{ \frac{Q}{\Delta} \left(dx + p^2\, dy \right)^2 - \frac{\Delta}{Q}\, dq^2 + \frac{\Delta}{P}\, dp^2 + \frac{P}{\Delta} \left(dx - q^2 dy \right)^2 \right\} . \tag{4.3}
$$

with

$$
\begin{aligned}
H(p,q) &:= 1 - \mu p q \,, \\
P(p) &:= b - g^2 + 2np - \varepsilon p^2 + 2m\mu\, p^3 + \left(-\frac{\lambda}{3} - \mu^2 b - \mu^2 e^2 \right) p^4 \,, \\
Q(q) &:= -(b + e^2) + 2mq - \varepsilon q^2 + 2n\mu\, q^3 + \left(\frac{\lambda}{3} + \mu^2 b - \mu^2 g^2 \right) q^4 \,, \\
\Delta(p,q) &:= p^2 + q^2 \,,
\end{aligned} \tag{4.4}
$$

while the nonvanishing electromagnetic field components are given by

$$
\begin{aligned}
F_{xq} &:= \frac{(p^2 - q^2)e - 2gpq}{2\tilde{\Delta}^2} \,, \\
F_{yq} &:= p^2 F_{xq} \,, \\
F_{xp} &:= \frac{g(q^2 - p^2) - 2epq}{\tilde{\Delta}^2} \,, \\
F_{yp} &:= -q^2 F_{xp} \,.
\end{aligned} \tag{4.5}
$$

In order to express our solutions (4.4) and (4.5) in terms of the advanced and retarded time coordinates $\widetilde{u}$, $\widetilde{v}$, one has to use the inversion procedure,presented in the previous section, yielding $p = p(u,v)$ and $q = q(u,v)$. The explicit representation requires the use of tables for the sn function.

Our class of solutions, defined in region I, can be extended to the full spacetime by introducing the Heaviside step function

$$
\Theta(x) = \begin{cases} 1\,, & x \geq 0 \\ 0\,, & x < 0 \end{cases} \,, \tag{4.6}
$$

with $\Theta^2(x) = \Theta(x)$, and performing the following transformation

$$
u \to \Theta(u) u \,, \qquad \text{and} \qquad v \to \Theta(v) v \,. \tag{4.7}
$$

For the case of real roots of the polynomials, treated extensively in Sec. III.B, we obtain for region II

$$p(u,v) = \frac{p_1 p_{42} - p_2 p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}\widetilde{v}\right)^2}{p_{42} - p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}\widetilde{v}\right)^2}\,, \tag{4.8}$$

$$q(u,v) = \frac{q_1 q_{42} - q_2 q_{41}\left(\mathrm{sn}\frac{1}{2\mu_\theta}\widetilde{v}\right)^2}{q_{42} - q_{41}\left(\mathrm{sn}\frac{1}{2\mu_\theta}\widetilde{v}\right)^2}\,, \tag{4.9}$$

where

$$\mathrm{sn}\frac{\widetilde{v}}{2\mu} = \mathrm{sn}\left[\frac{1}{\mu}\arcsin v\right]\,. \tag{4.10}$$

For region III

$$p(u,v) = \frac{p_1 p_{42} - p_2 p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}\widetilde{u}\right)^2}{p_{42} - p_{41}\left(\mathrm{sn}\frac{1}{2\mu_\phi}\widetilde{u}\right)^2}\,, \tag{4.11}$$

$$q(u,v) = \frac{q_1 q_{42} - q_2 q_{41}\left(\mathrm{sn}\frac{1}{2\mu_\theta}\widetilde{u}\right)^2}{q_{42} - q_{41}\left(\mathrm{sn}\frac{1}{2\mu_\theta}\widetilde{u}\right)^2}\,, \tag{4.12}$$

where

$$\mathrm{sn}\frac{\widetilde{u}}{2\mu} = \mathrm{sn}\left[\frac{1}{\mu}\arcsin u\right]\,. \tag{4.13}$$

At this point it is important to mention that not every cylindrically symmetric spacetime, even in vacuum, satifies the requirements of Ernst [8] for being interpreted as colliding waves, only certain classes of cylindrically symmetric solutions can be thought of as solutions generated by collision of waves.

Moreover, up to here our analysis has been concerned mostly with metrical aspects leaving aside the task of having suitable field and matter energy–momentum tensors. As it is well known, each participating wave in the head–on collision is assumed to be plane fronted, i.e. it is characterized by a covariantly constant null eigenvector k, i.e., $k_{\mu;\nu} = 0$, of the Weyl Petrov type N conformal tensor. This condition implies a null energy–momentum tensor for the electromagnetic field, i.e. a radiation field or electrovacuum, —let us call it null field for short— into the Einstein field equations, c.f. Kramer et. al. [16], Section §21.5. Therefore, the existence of colliding wave solutions with non–vanishing cosmological λ term is thus forbidden.

Usually one derives colliding wave solutions using the backward way procedure, namely, one starts with a cylindrically symmetric solution and an energy–momentum tensor in terms of fields depending on the usual two null variables, say u and v, restricted to a closed interaction region, region I in our approach. Furthermore, by accomplishing on it the transformations (4.7) with Heaviside functions one obtains the spacetimes prior to the collision. These spacetimes occur to be type N solutions to the Einstein equations in vacuum or coupled with null fields, regions II and III in our approach, where the gravitational and null fields depend solely on a single variable (u or v). At this level, the colliding wave condition allows only for fields which can be matched to the flat Minkowski spacetime (region IV) background where the wavefronts propagate [1]. In other words, if the cylindrically symmetric spacetime is endowed with a cosmological λ term, one has to set it equal to zero in order to connect smoothly the different regions of the collision and in order to avoid meaningless jumps of the cosmological constant across the null lines.

On the other hand, it is worthwhile to mention the results by Chandrasekhar and Xanthopoulos [17] concerning colliding wave solutions with a perfect fluid (stiff matter, fluid pressure = fluid energy density). They found that after the collision takes place, the incoming plane waves supporting radiative fields, give rise to solutions with stiff matter perfect fluid tensor. It is an example of the transformation of massless particles describing null trajectories into a perfect fluid in which the stream lines describe timelike trajectories.

In full, in order to interpret our class of cylindrically symmetric solutions (4.4) as a colliding waves class of solutions, it is compulsory to set $\lambda = 0$ on it [16].

V. DISCUSSION

In this paper we present the formulation of the most general class nowadays of colliding waves with fourth degree polynomials determining the non–killingian sector, by means of a

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[1] We appreciate the referee's comment which draw our attention to this respect.

generalization of the advanced and retarded coordinates concept. As it has been pointed out, after setting the cosmological λ term equal to zero, this class of solutions describes the scattering of two noncolinear polarized gravitation plane waves, having a null electromagnetic field as a source. At the leading edge of each colliding type–N gravitational wave, the curvature tensor exhibits delta and jump discontinuities. The former is interpreted as a gravitational impulsive wave, whereas the latter is attributed to a gravitational shock wave.

As stated above, this class of solutions, defined again in region I, can also be extended to the full spacetime by introducing the Heaviside step function Θ.

The electromagnetic field present delta singularities and jump discontinuities. However, the Bianchi identities hold in a distributional sense, see [18].

There are no problems on the right–hand side because the delta type singularities of the curvature are multiplied by the smooth distributions $\sqrt{1-\Theta(u)u^2}$ and $\sqrt{1-\Theta(v)v^2}$, respectively.

On the other hand, with respect to the singularities of the solution, we have in Region I, the solution is of type D in the classification of Petrov, this is so because the only novanishing Weyl scalar is Ψ_2, and explicitly, it is given by

$$\begin{aligned}\Psi_2 = \frac{H^3}{2\Delta^3}&[2(e^2+g^2)(1+\mu pq)(p-iq)^2 \\ &+q(q^2-3p^2)(a_3-ia_1)-p(p^2-3q^2)(a_1+ia_3)]\,,\end{aligned} \tag{5.1}$$

where the parameters $a_1 = 2n, a_3 = 2m$. The parameters ϵ and b do not appear in the curvature scalar, however the properties of the solution depend on them, as we shall see. The complex structure of Ψ_2 is due to the noncolinear polarization of the incoming waves (nondiagonal metric, see [19]). Since we have the explicit expression for the Weyl scalar, Eq. (5.1), we can study the behavior of the invariants of the spacetime and determine if the solution develops a curvature singularity a finite time after the instant of collision. Penrose attributed the development of the singularity in most colliding wave solutions to the self-focusing of the gravitational waves on scattering. In this case the invariants are given by $I = 3\Psi_2^2$ and $J = \Psi_2^3$. Then the singularities of Ψ_2 are clearly the same for the invariants. A

singularity occurs when $\tilde{\Delta} = p^2 + q^2 = 0$, i. e. when simultaneously $p = 0$ and $q = 0$. The interesting question is if there is any case in which the nature of the roots do not permit that simultaneously $p = 0, q = 0$. If this is the case, then the solution is free of the focusing singularity. A brief analysis shows that this occurrence can be avoided for certain values of the parameters of the solution.

Let us consider the expressions given in Sec III.C for the case of P and Q both with four real roots. In this case (3.53) and (3.54) express p and q in terms of (u, v). Analyzing those expressions we see that $p = 0$, $q = 0$ simultaneously if, for instance, $p_{42} = 0, p_{41} = 0$ and $q_{42} = 0$, $q_{41} = 0$, i. e. when at least three of the four roots coincide: $p_4 = p_2 = p_1$ and also $q_4 = q_2 = q_1$. The assumption of the occurrence of three equal roots for P and for Q leads to algebraic conditions that must be satisfied by the parameters of the solution, $b, g, n, m, \varepsilon, e$ and μ. Some of these conditions are incompatible. For instance, in the case considered, the following condition must be valid: $\mu^2(b + e^2) = \mu^2(b - g^2)$. This is only true in the case if $e^2 + g^2 = 0$; i. e. if $e^2 + g^2 \neq 0$ we do not have that $p = 0$ and $q = 0$ simultaneously and then the singularity fails to occur. Then we conclude that in some cases the existence of the electromagnetic field permits to tune the parameters in such a way to avoid the focusing singularity. A related result was obtained already by Chandrasekhar for a colliding wave solution with a scalar field [20] and by García et al. in the framework of metric–affine gauge theories of gravity [21,22].

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